Absolute and Relative Risk Aversion, Stochastic Dominance

Econ 3030 Fall 2025

Lecture 12

Outline

1. Relative Risk Aversion
2. Absolute Risk Aversion
3. Stochastic Dominance
4. Expected Utility of Consumption

Summary from Last Class

- $F : \mathbb{R} \to [0, 1]$ is a cumulative distribution function; μ_F is the expected value of F.
- Preferences $\succsim$ are over the space of all cumulative distribution functions.
- There exists a utility function U on distributions defined as $U(F) = \int v\, dF$, for some continuous index $v : \mathbb{R} \to \mathbb{R}$ over wealth, such that: $F \succsim G \Leftrightarrow \int v\, dF \geq \int v\, dG$
  - The vNM utility index $v : \mathbb{R} \to \mathbb{R}$ is defined over wealth.
- $\succsim$ is risk averse if, for all cumulative distribution functions F, $\delta_{\mu_F} \succsim F$.
- The certainty equivalent (CE) of F, denoted $c(F, v)$, is defined by $v(c(F, v)) = \int v(\cdot)\, dF = U(F)$.
- The risk premium of F, denoted $r(F, v)$ is defined by $r(F, v) = \mu_F - c(F, v)$.
- Result:

$$\succsim \text{ is risk averse} \Leftrightarrow v \text{ is concave} \Leftrightarrow r(F, v) \geq 0$$

Picture

Relative Risk Aversion

- When can we say that one decision maker is more risk averse than another?
- Relative risk aversion answers this question.

Definition

Given two preference relations, $\succsim_1$ is more risk averse than $\succsim_2$ if and only if

$$F \succsim_1 \delta_x \qquad \Rightarrow \qquad F \succsim_2 \delta_x$$

for all F and x.

- If DM1 prefers the lottery F to receiving x for sure, then anyone who is less risk averse than DM1 also prefers the lottery F to receiving x for sure.
- Conversely, if DM2 prefers receiving x for sure to the lottery F, then anyone who is more risk averse than DM2 also prefers receiving x for sure to the lottery F.
- Again, this definition does not assume anything about preferences.
  - When both preferences satisfy expected utility, we have extra implications.

Relative Risk Aversion

- Relative risk aversion is equivalent to: "more concavity" of the utility index, a smaller certainty equivalent, and a larger risk premium.

Proposition

Suppose $\succsim_1$ and $\succsim_2$ are preference relations represented by the vNM indices v_1 and v_2. The following are equivalent:

1. $\succsim_1$ *is more risk averse than* $\succsim_2$*;*
2. $v_1 = \phi \circ v_2$ *for some strictly increasing concave* $\phi : \mathbb{R} \to \mathbb{R}$*;*
3. $c(F, v_1) \leq c(F, v_2)$*, for all* F*;*
4. $r(F, v_1) \geq r(F, v_2)$*, for all* F*.*

Proof.

Question 5 in Problem Set 6 □

An Application: Asset Demand

- An asset is a divisible claim to a financial return in the future.

Asset Demand

- An agent has initial wealth w; she can invest it either in a safe asset that returns \$1 per dollar invested, or in a risky asset that returns \$$z$ per dollar invested (where z is random).
  - The general version has N assets each yielding a return z_n per unit invested.
- The risky return has cdf $F(z)$, and assume $\int z\,dF > 1$.
  - What does this mean?
- Let α and β be the amounts invested in the risky and safe asset respectively.
  - One can think of (α, β) as a portfolio allocation that pays $\alpha z + \beta$.
- The agent solves

$$\max \int v(\alpha z + \beta)\,dF \quad \text{s. t.} \quad \alpha, \beta \geq 0 \text{ and } \alpha + \beta = w$$

  - What are the choice variables?

Optimal Portfolio Choice

- An asset is a divisible claim to a financial return in the future.

Optimal Portfolio Choice

- The agent solves

$$\max_{\alpha,\beta} \int v(\alpha z + \beta)\, dF \quad \text{s. t.} \quad \alpha, \beta \geq 0 \text{ and } \alpha + \beta = w$$

  - or $\max_{\alpha,\beta} \int v(\alpha z + w - \alpha)\, dF \quad \text{s. t.} \quad \alpha, \beta \geq 0$
- The first oder conditions for this optimal portfolio problem is

$$\int (z-1)v'(\alpha(z-1) + w)\, dF = 0$$

- If the decision maker is risk averse, this expression is decreasing in α
  - this follows from the concavity of v.
- One can use this fact to verify that if DM1 is more risk averse than DM2 then her optimal α_1 is smaller than the corresponding α_2:
  - the more risk averse consumer invests less in the risky asset.

How to Measure Risk Aversion

- Since concavity of v reflects risk aversion, v'' is a natural candidate measure of risk aversion.
- Unfortunately, v'' is not appropriate since it is not robust to strictly increasing linear transformations.

Definition

Suppose $\succsim$ is a preference relation represented by the twice differentiable vNM index $v : \mathbb{R} \to \mathbb{R}$. The Arrow–Pratt measure of absolute risk aversion $\lambda : \mathbb{R} \to \mathbb{R}$ is defined by

$$\lambda(x) = -\frac{v''(x)}{v'(x)}.$$

- The second derivative is normalized to measure risk aversion properly.
- Notice that by integrating $\lambda(x)$ twice one could recover the utility function.
  - How about the constants of integration?

Absolute Risk Aversion

Proposition

Suppose $\succsim_1$ and $\succsim_2$ are expected utility preference relations represented by the twice differentiable vNM indices v_1 and v_2. Then

$$\succsim_1 \text{ is more risk averse than } \succsim_2 \quad \Leftrightarrow \quad \lambda_1(x) \geq \lambda_2(x) \text{ for all } x \in \mathbb{R}$$

- This confirms that the Arrow-Pratt coefficient is the correct measure of increasing absolute risk aversion.

$$\succsim_1 \textit{ is more risk averse than } \succsim_2 \quad \Leftrightarrow \quad \lambda_1(x) \geq \lambda_2(x) \textit{ for all } x \in \mathbb{R}$$

Proof.

We know $v_1 = \phi(v_2)$ for some strictly increasing ϕ (by the homework).

- Differentiating
$$v_1'(x) = \phi'(v_2(x))v_2'(x) \quad \text{and} \quad v_1''(x) = \phi'(v_2(x))v_2''(x) + \phi''(v_2(x))(v_2'(x))^2$$
- Dividing v_1'' by $v_1' > 0$ we have $\frac{v_1''(x)}{v_1'(x)} = \frac{\phi'(v_2(x))v_2''(x)}{v_1'(x)} + \frac{\phi''(v_2(x))(v_2'(x))^2}{v_1'(x)}$
- From the first equation $\phi'(v_2(x)) = \frac{v_1'(x)}{v_2'(x)}$ so
$$\frac{v_1''(x)}{v_1'(x)} = \frac{v_2''(x)}{v_2'(x)} + \frac{\phi''(v_2(x))(v_2'(x))^2}{v_1'(x)} \text{ or } -\frac{v_1''(x)}{v_1'(x)} = -\frac{v_2''(x)}{v_2'(x)} - \frac{\phi''(v_2(x))(v_2'(x))^2}{v_1'(x)}$$
- using the definition of Arrow-Pratt:
$$\lambda_1(x) = \lambda_2(x) + \textit{something positive},$$
if and only if ϕ is concave (since v is increasing). □

First Order Stochastic Dominance

Suppose we do not know the decision maker's utility index v.

- We know is that it is increasing.
- We do not know how she will rank all lotteries, but we know how she will rank a particular subset.

First Order Stochastic Dominance (FOSD)

- All that is known about a decision maker is that she likes more wealth rather than less wealth.
- We can deduce how she ranks some lotteries using the following transitive, but incomplete, binary relation.

Definition

F first-order stochastically dominates G, denoted $F \succeq_{\text{FOSD}} G$, if

$$\int v\, dF \geq \int v\, dG,$$

for every nondecreasing function $v : \mathbb{R} \to \mathbb{R}$.

- If $F \succeq_{\text{FOSD}} G$, then anyone who prefers more money to less prefers F to G.
  - This follows because $F \succsim G \Leftrightarrow U(F) = \int v\, dF \geq \int v\, dG = U(G)$.
- Why do we care about this?
- Because if $F \succeq_{\text{FOSD}} G$ we can conclude that anyone who likes more money would choose F over G regardless of what their actual function v looks like.

First Order Stochastic Dominance (FOSD)

Definition

F first-order stochastically dominates G, denoted $F \succeq_{\text{FOSD}} G$, if

$$\int v\, dF \geq \int v\, dG,$$

for every nondecreasing function $v : \mathbb{R} \to \mathbb{R}$.

- FOSD is characterized by comparing cumulative distribution functions.

Proposition

$F \succeq_{FOSD} G$ if and only if $F(x) \leq G(x)$ for all $x \in \mathbb{R}$.

Second Order Stochastic Dominance

Suppose we do not know the decision maker's utility index v.

- We know is that it is increasing.
- We also know that the decision maker is risk-averse.

- We do not know how she will rank all lotteries, but we know how she will rank a particular subset.

Second Order Stochastic Dominance (SOSD)

If one also knows that the decision maker is risk averse (her utility index for wealth is concave), we know how she ranks more cumulative distributions functions.

Definition

F second-order stochastically dominates G, denoted $F \succeq_{\text{SOSD}} G$, if

$$\int v \, dF \geq \int v \, dG,$$

for every nondecreasing *concave* function $v : \mathbb{R} \to \mathbb{R}$.

- $F \succeq_{\text{SOSD}} G$ means a DM who likes more money and is risk-averse prefers F to G.
- $\succeq_{\text{SOSD}}$ ranks distributions that are not necessarily ranked by $\succeq_{\text{FOSD}}$.

Second Order Stochastic Dominance (SOSD)

Definition

F second-order stochastically dominates G, denoted $F \succeq_{\text{SOSD}} G$, if

$$\int v \, dF \geq \int v \, dG,$$

for every nondecreasing *concave* function $v : \mathbb{R} \to \mathbb{R}$.

- SOSD is also characterized by looking at cumulative distribution functions.

Proposition

$F \succeq_{SOSD} G$ if and only if $\int_{-\infty}^{x} F(t) \, dt \leq \int_{-\infty}^{x} G(t) \, dt$ for all $x \in \mathbb{R}$.

Lotteries Over Consumption Bundles

Question

- How can we connect preferences over random consumption bundles to preferences over random amounts of money?

- There are n commodities; $\succsim$ ranks simple lotteries on $X = \mathbb{R}^n_+$.
- Let $U : X \to \mathbb{R}$ be the expected utility function representing $\succsim$

$$\pi \succ \rho \Leftrightarrow U(\pi) > U(\rho)$$

- Fix prices $\mathbf{p} \in \mathbb{R}^n_{++}$, and assume they are known.
- $x^*(\mathbf{p}, w)$ is the corresponding Walrasian demand, and
- $v(\mathbf{p}, w)$ is the induced indirect utility function ($v(\mathbf{p}, w) = U(\mathbf{x}^*)$ with $\mathbf{x}^* \in x^*(\mathbf{p}, w)$).

Preferences and Lotteries Over Money

- Let $w \in [0, \infty)$ be the consumer's uncertain income

How does the consumer rank lotteries over income?

- A lottery over income $\pi(w)$ is a probability distributions on $[0, \infty)$.
- The expected utility of π is
$$\sum\nolimits_{w \in support(\pi)} \pi(w)\, v(\mathbf{p}, w)$$
where $v(\mathbf{p}, w) = U(\mathbf{x}^*)$ with $\mathbf{x}^* \in x^*(\mathbf{p}, w)$
- Thus, the consumer ranks lotteries over income as follows
$$\pi \succ \rho \Leftrightarrow \sum_{w \in support(\pi)} \pi(w)\, v(\mathbf{p}, w) > \sum_{w \in support(\pi)} \rho(w)\, v(\mathbf{p}, w)$$
- The indirect utility function $v(\cdot)$ is the vNM utility of income.
- Preferences over lottteries are induced by preferences over consumption bundles.

- Think carefully about the timing.

Indirect Uility and Lotteries Over Money

- What is the formal connection between preferences over consumption bundles (represented by U) and expected utility over money that uses the vNM function v?

Proposition

Assume $\succsim$ over lotteries on X satisfy the vonNeuman & Morgenstern axioms; let $U : X \to \mathbb{R}$ be the expected utility function representing $\succsim$, and let $v(\mathbf{p}, w)$ be the induced indirect utility function. Then, the consumer preferences over lotteries on w also satisfy the vonNeuman & Morgenstern axioms and the function $w \to v(\mathbf{p}, w)$ is the consumer's utility function. Moreover:

1. *$v(\mathbf{p}, w)$ is continuos in w.*
2. *If U is locally nonsatiated, then $v(\mathbf{p}, w)$ is strictly increasing in w.*
3. *If U is a concave, then $v(\mathbf{p}, w)$ is concave in w (the consumer is risk-averse).*

- Note that concavity/risk-aversion here says that a consumer prefers the bundle $\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{x}'$ to the lottery that gives $\mathbf{x}$ with probability 0.5 and $\mathbf{x}'$ with probability 0.5.

Next Class

- Beyond Expected Utility